

Axially symmetric Einstein-Straus models

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The existence of static and axially symmetric regions in a Friedman-Lemaître (FL) cosmology is investigated under the only assumption that the cosmic time and the static time match properly on the boundary hypersurface. It turns out that the most general form for the static region is a two-sphere with arbitrarily changing radius which moves along the axis of symmetry in a determined way. The geometry of the interior region is completely determined in terms of background objects. When any of the most widely used energy-momentum contents for the interior region is imposed, both the interior geometry and the shape of the static region must become exactly spherically symmetric. This shows that the Einstein-Straus model, which is the generally accepted answer for the null influence of the cosmic expansion on the local physics, is not a robust model and it is rather an exceptional and isolated situation. Hence, its suitability for solving the interplay between cosmic expansion and local physics is doubtful and more adequate models should be investigated.

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I. INTRODUCTION

The influence of the large scale geometry of the Universe on the local physics around astrophysical objects is a very old and fundamental question in gravitational physics. The fact that the geometry of the Universe as a whole can be approximated very accurately by an expanding Friedman-Lemaître (FL) model leads naturally to consider whether the cosmological expansion has any effect on the local physics at astrophysical scales. This type of question was probably first addressed by McVittie [1] who used a spherically symmetric spacetime supposedly describing a point particle immersed in a Friedman-Lemaître background (see, however, [2]). This metric was later analyzed in more detail by Järnefeld [3]. Both authors concluded that the effect of the cosmological expansion on the planetary orbits is very small and with no measurable consequences. More recently, Gautreau [4] analyzed the problem by using an inhomogeneous and spherically symmetric model in the so-called curvature coordinates [5] (see [6] for a review of these results). However, the generally accepted answer to this problem was given by Einstein and Straus in [7] where a model describing a massive point particle surrounded by a spherically symmetric vacuum region embedded in a dust FL cosmology was presented (the now so-called Einstein-Straus model). The boundary of the vacuum region is a two-sphere comoving with the cosmological flow. Thus, Einstein and Straus showed the possibility of having static regions embedded in an expanding cosmological background thus implying that the cosmic expansion does not influence the local physics around the massive object. Despite its clear implications and simplicity, Einstein and Straus' construction suffers from several important limitations. First, the interior geometry of the static region is exactly spherically symmetric and its boundary is a two-sphere comoving with the cosmological flow. This a very strong assumption, specially regarding the shape of the static cavity. Second, given the dust background metric, the radius of the vacuum region surrounding the compact object is uniquely fixed by its mass. This implies that some situations of astrophysical interest are not satisfactorily described by the Einstein-Straus model. Furthermore, Krasiński has argued [6] that the model is unstable, meaning that a matching between the vacuum cavity and the FL background with a slightly bigger radius than the Einstein-Straus value makes the static region expand with respect to the cosmological flow (and vice versa for a smaller radius).

Hence, more general models should be analyzed in order to reach a definitive conclusion on whether the influence of the cosmological expansion on the local physics is indeed vanishing and, in that case, on the suitability of the Einstein and Straus' construction to provide a completely satisfactory explanation. In order to test the importance of the spherically symmetric assumption, other geometries for the interior region (especially the shape of the boundary) should be considered. Since the local geometry is generally believed to be static (which is the translation of the null influence of the background expansion), the problem can be formulated as to whether non-spherical static cavities can be embedded in a FL cosmology. In a very recent paper, Senovilla and Vera [8] proved the surprising fact that embedding a cylindrically symmetric static region in an expanding FL cosmology is *always* impossible (irrespective of the matter contents inside the cavity). This result supports the view that spherical symmetry may indeed be an essential ingredient for the existence of the static region. In order to clarify this question, we address the existence of static cavities in a FL background in a rather general situation. The essential simplifying assumption we make (otherwise the problem becomes formidable) is that the interior static region is axially symmetric, both in shape and in spacetime geometry (no restrictions on the interior energy-momentum contents are imposed). This assumption is physically very reasonable because axial symmetry is likely to be present in many realistic situations. The second assumption we make is that the two canonical time functions in each region (the cosmic time in the FL background and the static time in the interior region) match properly across the boundary of the static region, so that the global spacetime has a well-defined and natural time function. This model is general enough to include both spherical symmetry and cylindrical symmetry as particular cases and, hence, its analysis will clarify to what extent the extreme situations in which the matching is possible (spherical symmetry) and never feasible (cylindrical symmetry) are generic.

Considering static regions embedded in an expanding background requires the use of the theory of matching between spacetimes. This consists in imposing the *matching conditions* which can be viewed as a set of partial differential equations to be solved. When the symmetry of the matching problem is high enough (i.e. when the two spacetimes to be matched possess at least two isometries which are

preserved by the matching hypersurface), the equations become ordinary differential equations and the problem simplifies considerably. Most of the explicit matchings which have been performed in the literature allow for this type of reduction. In particular, this is true for the Einstein-Straus model and the cylindrically symmetric case described above. In our case, however, the problem is technically more complex because the global spacetime possesses *only* one isometry, namely axial symmetry. In order to solve it, we use a set of *constraint matching conditions*, which are necessary consequences of the matching conditions on any spacelike two-dimensional surface embedded into the matching hypersurface. They appear naturally when developing a $2+1$ geometrical decomposition of the matching conditions into a set of constraints on initial spacelike two-surfaces and a set of evolution equations determining how the two-surface must evolve in order to generate the three-dimensional matching hypersurface. The full decomposition has not been completed yet and will be reported elsewhere. For the purposes of this paper it will suffice exploiting the set of constraint matching conditions.

The paper is organised as follows. In section 2, the matching theory between spacetimes is briefly reviewed and the constraint matching conditions on spacelike two-dimensional surfaces are presented. In section 3, the matching problem between a static and axially symmetric spacetime and a Friedman-Lemaître expanding cosmology is solved. Since the problem is technically non-trivial, the procedure is described in some detail. First, the constraint matching conditions are imposed. They restrict the form of the matching hypersurface severely. It turns out that the most general matching hypersurface consists of a collection of two-spheres, two-planes or hyperbolic planes centred on the axis of symmetry and moving arbitrarily along this axis. For closed FL models only two-spheres are possible; for flat FL models both two-spheres and two-planes are allowed while for the open case all three cases are possible. We restrict the analysis to the two-sphere case because this is the only one describing a spatially compact region embedded in a FL cosmology. Then, the remaining set of matching conditions are imposed. They determine the interior geometry of the static region in terms of the background FL metric and the function $r(t)$ describing how the radius of the two-spheres generating the matching hypersurface varies with time. The motion of the set of two-spheres along the axis of symmetry is also determined by the background metric and $r(t)$. In section 4, a detailed study of the static and axially symmetric interior geometry is performed and a number of important conclusions are obtained. In particular, the interior metric is shown to be nearly spherically symmetric with both its energy-momentum tensor and Petrov type of the same type as in the exact spherically symmetric case. Furthermore, analyzing the most widely used energy-momentum tensors for the interior region (vacuum, Λ -term, perfect fluid and electrovacuum) we show that the interior geometry must be *exactly* spherically symmetric in all those cases and the boundary of the cavity must be generated by a two-sphere with its centre being at rest with respect to the cosmological flow. Hence, we conclude that the Einstein-Straus model is essentially unique (when axial symmetry is imposed) and that the spherical symmetry assumption is indeed crucial for the existence of static cavities inside a FL cosmology. Consequently, obtaining a robust model determining whether or not the cosmological expansion has any influence on the local physics is still an open problem which deserves further analysis and investigation.

II. JUNCTION OF SPACETIMES AND CONSTRAINT MATCHING CONDITIONS.

The matching (see [9] and references therein) between two spacetimes with boundary, (V^+, g^+) and (V^-, g^-) with corresponding boundaries Ω^+ and Ω^- , consists in constructing a spacetime manifold (V, g) by identifying the points, (and the tangent spaces) on the two boundaries, which are therefore called *matching hypersurfaces*. The point-to-point identification requires the existence of a diffeomorphism between Ω^+ and Ω^- . Therefore, they can be also considered as diffeomorphic to an abstract three-dimensional manifold Ω so that there exist two embeddings

$$\Phi^+ : \Omega \longrightarrow V^+, \quad \Phi^- : \Omega \longrightarrow V^-,$$

satisfying $\Phi^+(\Omega) = \Omega^+$ and $\Phi^-(\Omega) = \Omega^-$. Using local coordinates $\{\xi^i\}$ for Ω ($i, j, \dots = 1, 2, 3$), $\{x_+^\alpha\}$ for V^+ ($\alpha, \beta, \dots = 0, 1, 2, 3$) and $\{x_-^\alpha\}$ for V^- the two embeddings take a local form

$$\Phi^+ : \xi^i \longrightarrow x_+^\alpha(\xi), \quad \Phi^- : \xi^i \longrightarrow x_-^\alpha(\xi),$$

where $x_+^\alpha(\xi)$ and $x_-^\alpha(\xi)$ are functions of ξ^i . As an important theorem by Clarke and Dray [10] shows, the construction of a spacetime $V = V^- \cup V^+$ with continuous metric g can be performed if and only if the two first fundamental forms $\Phi^{+*}(g^+)$ and $\Phi^{-*}(g^-)$ on Ω coincide (Φ^* denotes the pull-back of the map Φ). In local coordinates, these conditions read

$$g_{\alpha\beta}^+ \Big|_{x_+(\xi)} \frac{\partial x_+^\alpha}{\partial \xi^i} \frac{\partial x_+^\beta}{\partial \xi^j} = g_{\alpha\beta}^- \Big|_{x_-(\xi)} \frac{\partial x_-^\alpha}{\partial \xi^i} \frac{\partial x_-^\beta}{\partial \xi^j}, \quad (1)$$

and will be called *preliminary matching conditions*. When (1) is fulfilled, the Riemann tensor in (V, g) is well-defined in a distributional sense. In general, this distribution has a Dirac delta part with support on the matching hypersurface, which can be used, when appropriate, to describe thin shells of matter and/or pure gravitational field. Whenever these physical objects are absent, the Dirac delta contribution must vanish and the *second set of matching conditions* must be imposed. First, two transverse vector fields, \vec{l}_+ on Ω^+ and \vec{l}_- on Ω^- , (called *rigging* vector fields) must be chosen. As they prescribe how the tangent spaces are to be identified, their choice must be compatible with the continuity of the metric in V . This demands

$$\begin{aligned} g_{\alpha\beta}^+ l_+^\alpha l_+^\beta \Big|_{x_+(\xi)} &= g_{\alpha\beta}^- l_-^\alpha l_-^\beta \Big|_{x_-(\xi)}, \\ g_{\alpha\beta}^+ l_+^\alpha \Big|_{x_+(\xi)} \frac{\partial x_+^\beta}{\partial \xi^j} &= g_{\alpha\beta}^- l_-^\alpha \Big|_{x_-(\xi)} \frac{\partial x_-^\beta}{\partial \xi^j}. \end{aligned} \quad (2)$$

The two riggings \vec{l}_+ and \vec{l}_- must have different relative orientations with respect to Ω^+ and Ω^- , respectively (i.e. either \vec{l}_+ points outside V^+ everywhere and \vec{l}_- points inside V^- everywhere or vice versa). Once the riggings are chosen, the matching conditions demand that the pull-back of the covariant derivative of the rigging one-form (obtained by lowering the index to \vec{l}) coincides when calculated using any of both embeddings. In local coordinates,

$$\begin{aligned} l_{+\beta} \Big|_{x_+(\xi)} \frac{\partial x_+^\alpha}{\partial \xi^i} \nabla_+^\alpha \left(\frac{\partial x_+^\beta}{\partial \xi^j} \right) &= l_{-\beta} \Big|_{x_-(\xi)} \frac{\partial x_-^\alpha}{\partial \xi^i} \nabla_-^\alpha \left(\frac{\partial x_-^\beta}{\partial \xi^j} \right) \iff \\ l_{+\beta} \Big|_{x_+(\xi)} \left(\frac{\partial^2 x_+^\beta}{\partial \xi^i \partial \xi^j} + \Gamma_{\mu\nu}^{+\beta} \frac{\partial x_+^\mu}{\partial \xi^i} \frac{\partial x_+^\nu}{\partial \xi^j} \right) &= l_{-\beta} \Big|_{x_-(\xi)} \left(\frac{\partial^2 x_-^\beta}{\partial \xi^i \partial \xi^j} + \Gamma_{\mu\nu}^{-\beta} \frac{\partial x_-^\mu}{\partial \xi^i} \frac{\partial x_-^\nu}{\partial \xi^j} \right), \end{aligned} \quad (3)$$

where ∇^+ (∇^-) denotes the covariant derivative and $\Gamma_{\mu\nu}^{+\beta}$ ($\Gamma_{\mu\nu}^{-\beta}$) are the corresponding Christoffel symbols in V^+ (V^-).

In many physically interesting problems the degree of symmetry is high enough so that the conditions (1) and (3) become a system of ordinary differential equations, which simplifies the problem considerably. In general, however, this reduction does not happen and the partial differential system must be considered. As we shall see below, some of these cases can be handled more easily by using a $2+1$ decomposition of the matching conditions. This consists in obtaining a set of geometrical conditions on an initial two-dimensional surface and evolution equations determining how this surface must evolve in order to generate the matching hypersurface. This theory will be reported elsewhere when fully developed. For the aim of this paper, only the *constraint matching conditions* will be necessary. In order to describe them, some notation must be introduced.

The main assumption for the $2+1$ decomposition is that the hypersurface Ω can be foliated by a set of two-dimensional surfaces, which will be denoted by $\{S_\tau\}$ (τ is a parameter identifying the element of the foliation and plays the role of an evolution parameter). Each two-surface in the foliation can be immersed into Ω by using the natural inclusion

$$i_\tau : S_\tau \longrightarrow \Omega.$$

The surface S_τ can then be embedded into V^+ and V^- , respectively, by the maps

$$\Phi_\tau^+ \equiv \Phi^+ \circ i_\tau, \quad \Phi_\tau^- \equiv \Phi^- \circ i_\tau \quad (4)$$

which define the two image surfaces $S_\tau^+ \equiv \Phi_\tau^+(S_\tau)$ and $S_\tau^- \equiv \Phi_\tau^-(S_\tau)$. We will further assume that both S_τ^+ and S_τ^- are *spacelike* everywhere. Let us consider a point $x \in S_\tau$ and define the vector space

$$(T_x S_\tau^+)_\perp \equiv \left\{ \mathbf{n} \in T_{\Phi_\tau^+(x)}^* V^+ ; \mathbf{n} (d\Phi_\tau^+(T_x S_\tau)) = 0 \right\},$$

where $d\Phi_\tau^+$ is the differential application of the embedding Φ_τ^+ , and $T_y M$, $T_y^* M$ denote, respectively, the tangent and cotangent spaces of a manifold M at any point $y \in M$. The vector space $(T_x S_\tau^+)_\perp$ is simply the set of normal one-forms to the two-surface S_τ^+ in V^+ . Since S_τ^+ is assumed to be spacelike, $(T_x S_\tau^+)_\perp$ is two-dimensional and timelike. The second fundamental form of S_τ^+ along the direction \mathbf{n} is defined, as usual, as

$$\mathbf{K}_{S_\tau}^+ (\mathbf{n}) \equiv \Phi_\tau^{+*} (\nabla^+ \mathbf{n}).$$

where \mathbf{n} is a one-form field orthogonal to S_τ^+ everywhere (the set of all these objects will be denoted by $(TS_\tau^+)_\perp$). Obviously, similar definitions hold by interchanging plus and minus everywhere.

The *constraint matching conditions* on S_τ are essentially the restrictions of the full set of matching conditions into the two-surface S_τ . They can also be splitted into two different sets, one involving first fundamental forms and another involving second fundamental forms. The first one is an immediate consequence of (1) and the definition (4) and demands, quite naturally, the equality of the two first fundamental forms on S_τ . Explicitly,

$$\Phi_\tau^{+*} (g^+) = \Phi_\tau^{-*} (g^-). \quad (5)$$

The second set of constraint matching conditions is somewhat more involved, although its geometrical contents can be clearly understood. The matching of the two spacetimes, requires, as mentioned above, the identification of the tangent spaces at the points on the matching hypersurfaces. Furthermore, this identification must be done so that the resulting metric tensor is continuous. When working with the foliation $\{S_\tau\}$, part of the job is accomplished by identifying the tangent planes to the two surfaces S_τ^+ and S_τ^- . The remaining identification requires the existence of a linear and isometric application

$$\xi_\tau^x : (T_x S_\tau^+)_\perp \longrightarrow (T_x S_\tau^-)_\perp, \quad (6)$$

which completes the identification of the tangent spaces at the points $\Phi_\tau^+(x)$ and $\Phi_\tau^-(x)$. It is not difficult to show that the preliminary and second set of matching conditions can be combined to imply the following condition on S_τ

$$\mathbf{K}_{S_\tau}^+ (\mathbf{n}) = \mathbf{K}_{S_\tau}^- (\xi_\tau (\mathbf{n})), \quad \forall \mathbf{n} \in (TS_\tau^+)_\perp, \quad (7)$$

where the field $\xi_\tau (\mathbf{n})$ is defined, naturally, as $\xi_\tau (\mathbf{n})|_{\Phi_\tau^-(x)} = \xi_\tau^x (\mathbf{n}|_{\Phi_\tau^+(x)})$, $\forall x \in S_\tau$. The existence of an isometric application (6) satisfying (7) constitutes the *second set of constraint matching conditions*. Due to the linearity of ξ_τ^x , and the \mathcal{F} -linearity of $\mathbf{K}_{S_\tau}^+$ and $\mathbf{K}_{S_\tau}^-$ (i.e. $\mathbf{K}_{S_\tau}^\pm (f\mathbf{n}) = f\mathbf{K}_{S_\tau}^\pm (\mathbf{n})$ for any scalar function f), it suffices to impose (7) for two one-form fields $\mathbf{n}_1, \mathbf{n}_2 \in (TS_\tau^+)_\perp$ being linearly independent at each point on S_τ^+ . Let us now write down the local form of the constraint matching conditions (5) and (7). Introducing local coordinates $\{\zeta^A\}$ ($A, B, \dots = 1, 2$) in the two-surface S_τ , the two embeddings read, locally,

$$\Phi_\tau^+ : \zeta^A \longrightarrow x_+^\alpha = x_\tau^{+\alpha} (\zeta), \quad \Phi_\tau^- : \zeta^A \longrightarrow x_-^\alpha = x_\tau^{-\alpha} (\zeta),$$

where $x_\tau^{+\alpha} (\zeta)$ and $x_\tau^{-\alpha} (\zeta)$ are functions of ζ^A . The condition (5) becomes, obviously,

$$g_{\alpha\beta}^+ \Big|_{x_\tau^+(\zeta)} \frac{\partial x_\tau^{+\alpha}}{\partial \zeta^A} \frac{\partial x_\tau^{+\alpha}}{\partial \zeta^B} = g_{\alpha\beta}^- \Big|_{x_\tau^-(\zeta)} \frac{\partial x_\tau^{-\alpha}}{\partial \zeta^A} \frac{\partial x_\tau^{-\alpha}}{\partial \zeta^B}. \quad (8)$$

Regarding (7), we take arbitrary bases, $\mathbf{n}_+^A|_{\Phi_\tau^+(x)}$ of $(T_x S_\tau^+)_\perp$ and $\mathbf{n}_-^A|_{\Phi_\tau^-(x)}$ of $(T_x S_\tau^-)_\perp$, and define the scalar quantities

$$\gamma_+^{AB}(x) \equiv g^+ \left(\mathbf{n}_+^A, \mathbf{n}_+^B \right) \Big|_{\Phi_\tau^+(x)}, \quad \gamma_-^{AB}(x) \equiv g^- \left(\mathbf{n}_-^A, \mathbf{n}_-^B \right) \Big|_{\Phi_\tau^-(x)}.$$

The linear isometric application ξ_τ^x takes the form

$$\xi_\tau^x \left(\mathbf{n}_+^A \Big|_{\Phi_\tau^+(x)} \right) = \xi_\tau^A(x) \mathbf{n}_-^B \Big|_{\Phi_\tau^-(x)},$$

where $\xi_\tau^A(x)$ are a set of scalar functions defined on S_τ which must satisfy

$$\xi_\tau^A(x) \xi_\tau^C(x) \gamma_-^{BD}(x) = \gamma_+^{AC}(x)$$

to ensure that ξ_τ^x is an isometry. The condition (7) can now be rewritten as

$$n_{+\beta}^C \left(\frac{\partial^2 x_\tau^{+\beta}}{\partial \zeta^A \partial \zeta^B} + \Gamma_{\mu\nu}^{+\beta} \frac{\partial x_\tau^{+\mu}}{\partial \zeta^A} \frac{\partial x_\tau^{+\nu}}{\partial \zeta^B} \right) \Big|_{x_\tau^+(\zeta)} = \xi_\tau^C(\zeta) n_{-\beta}^D \left(\frac{\partial^2 x_\tau^{-\beta}}{\partial \zeta^A \partial \zeta^B} + \Gamma_{\mu\nu}^{-\beta} \frac{\partial x_\tau^{-\mu}}{\partial \zeta^A} \frac{\partial x_\tau^{-\nu}}{\partial \zeta^B} \right) \Big|_{x_\tau^-(\zeta)}. \quad (9)$$

In the next section, the conditions (8) and (9) will be exploited to restrict the form of the matching hypersurface in the axially symmetric Einstein-Straus model. They will allow for a complete resolution of the matching problem we are considering.

III. AXIALLY SYMMETRIC EINSTEIN-STRAUS MODEL.

Let us now study the main object of this paper, namely the existence of static, axially symmetric regions embedded in a Friedman-Lemaître universe. We choose standard spherical coordinates in the FL cosmology so that the line-element takes the usual form

$$ds_{FL}^2 = -dt^2 + a^2(t) [dr^2 + \Sigma^2(r, \epsilon) (d\theta^2 + \sin^2 \theta d\phi^2)], \quad (10)$$

where $a(t)$ is the scale factor of the cosmological model (we do not restrict its form whatsoever a priori) and $\Sigma(r, \epsilon)$ is the standard function

$$\Sigma(r, \epsilon) = \begin{cases} \sinh r & \text{if } \epsilon = -1 \\ r & \text{if } \epsilon = 0 \\ \sin r & \text{if } \epsilon = 1 \end{cases}. \quad (11)$$

Regarding the static, axially symmetric spacetime, we will make the usual assumption that it is orthogonally transitive (i.e. the two-planes orthogonal to the isometry group orbits are surface-forming). As an important theorem by Kundt and Trümper [11] shows (generalizing Papapetrou's theorem for vacuum [12]), this condition is automatically satisfied for a wide class of energy-momentum tensors including vacuum, Λ -term (i.e. cosmological constant), perfect fluids and electrovacuum. Thus, the orthogonally transitive assumption does not represent a severe restriction on the kind of regions we are considering. Under this assumption, there exist coordinates in which the line-element in the static region reads

$$ds_{St}^2 = -F^2(x^A) dT^2 + g_{AB}(x^C) dx^A dx^B + W^2(x^A) d\hat{\phi}^2, \quad (12)$$

so that ∂_T is the static and $\partial_{\hat{\phi}}$ is the axial Killing vector. The coordinate transformations keeping the structure of (12) unchanged are

$$T \rightarrow \alpha_1 T + \alpha_0, \quad x^A \rightarrow x^A(\tilde{x}^A), \quad \hat{\phi} \rightarrow \hat{\phi} + c, \quad (13)$$

where $\alpha_1 \neq 0$, α_0 and c are arbitrary constants and x^A (\tilde{x}^A) are arbitrary functions with non-vanishing Jacobian.

Both FL and static spacetimes admit canonical foliations by spacelike hypersurfaces. In the FL cosmology, this foliation is defined by the homogeneity and isotropy hypersurfaces $t = \text{const}$. In the static region, the foliation is given by the hypersurfaces orthogonal to the static Killing vector, $T = \text{const}$. Since the physical problem we treat is the existence of static regions living in a cosmological background, we can assume that the matching hypersurfaces are not tangent to any of the hypersurfaces $t = \text{const}$. in the FL spacetime or $T = \text{const}$. in the static spacetime. This case would rather correspond to a phase transition between a static spacetime and an expanding FL cosmology, which is a completely different situation. Consequently, the foliation defined by the cosmic time induces a canonical foliation in the matching hypersurface Ω^{FL} , defined by $S_\tau^{FL} = \Omega^{FL} \cap \{t = \tau\}$. Similarly, the matching hypersurface in the static spacetime Ω^{St} (the labels *FL* and *St* will replace the scripts plus and minus used in the previous section) can be canonically foliated by the two-surfaces $S_T^{St} = \Omega^{St} \cap \{T = \mathcal{T}\}$. Hence, we have that each region in the glued spacetime has a natural and canonical time function, the cosmic time in the FL background and the static time in the interior region. It is natural from the physical point of view to demand that these two canonical times coincide on the matching hypersurface. Here, we make the second fundamental assumption in this paper, namely, we demand that the global Einstein-Straus spacetime possesses also a canonical time function which coincides with the cosmic time in the FL background and with the static time in the interior region¹. If this assumption were violated, then spacetime events on the boundary of the static region being equally old with respect to the cosmic time, would correspond to different instants of time with respect to the static time, which is an undesirable property for the model we are constructing. Notice that this condition we are imposing *is* satisfied in both the spherically symmetric Einstein-Straus model and in the static cylindrically symmetric model in [8]. Therefore, the assumption is not only physically plausible but it is also general enough so that a bridge between the two extreme cases is constructed. Hence, analyzing this case will certainly determine whether the spherically symmetric Einstein-Straus model is exceptional or not. It must be emphasized, however, that we are indeed making a simplifying assumption which is not mathematically necessary *a priori*. It would be interesting analyzing the problem in full generality, although this is probably a much more difficult question.

The condition we are imposing can be translated in mathematical terms by demanding that the two canonical foliations in the matching hypersurface coincide in the glued manifold. In other words, we assume there exists a foliation $\{S_\tau\}$ in the abstract matching hypersurface Ω such that the two embeddings defining the matching hypersurfaces

$$\Phi^{FL} : \Omega \longrightarrow V^{FL}, \quad \Phi^{St} : \Omega \longrightarrow V^{St},$$

satisfy $\Phi^{FL}(S_\tau) = S_\tau^{FL}$ and $\Phi^{St}(S_\tau) = S_{T(\tau)}^{St}$ (the geometrical coincidence of the foliation allows for a reparametrisation $T(\tau)$).

Regarding the condition that the static region is axially symmetric, this implies not only that the interior metric possesses an axial Killing vector, but also that the two matching hypersurfaces preserve this symmetry. More precisely, they must be invariant under the action of the axial symmetry in each spacetime. Since the two spacelike foliations in V^{FL} and V^{St} are also axially symmetric, it follows that each two-surface S_τ admits an axial Killing vector field. Choosing coordinates $\{\lambda, \varphi\}$ on S_τ adapted to this axial symmetry (i.e. such that the axial Killing reads ∂_φ), the most general form for the two embeddings of S_τ into V^{FL} and V^{St} can be easily seen to be

¹If the interior region possesses a high enough degree of symmetry (e.g. Minkowski spacetime) the static time may be not unique. However, since we did not specify the form of the interior metric (12) *a priori*, we are not choosing between any of them (whenever there is more than one) and therefore they are all treated on the same footing. In this particular case, the assumption we are making is that there exists *at least one* static time which matches appropriately with the cosmic time on the matching hypersurface. As we shall see below, the matching problem will fix the interior geometry completely and, therefore, it will choose which of the static times (if there is more than one) is the appropriate one to match with the exterior FL background.

$$\begin{aligned}\Phi_{\tau}^{FL} : S_{\tau} &\longrightarrow V^{FL} \\ \Phi_{\tau}^{FL} : \{\lambda, \varphi\} &\longrightarrow \{t = \tau, r = r_{\tau}(\lambda), \theta = \theta_{\tau}(\lambda), \phi = \varphi\},\end{aligned}\tag{14}$$

$$\begin{aligned}\Phi_{\tau}^{St} : S_{\tau} &\longrightarrow V^{St} \\ \Phi_{\tau}^{St} : \{\lambda, \varphi\} &\longrightarrow \left\{T = T(\tau), x^A = x_{\tau}^A(\lambda), \hat{\phi} = \varphi\right\},\end{aligned}\tag{15}$$

where $r_{\tau}(\lambda)$, $\theta_{\tau}(\lambda)$, $x_{\tau}^A(\lambda)$ and $T(\tau)$ are unknown functions. Imposing the first constraint matching conditions (5) gives the two relations

$$W(x^A) \Big|_{\Psi_{\tau}^{St}(x)} = a\Sigma \sin \theta \Big|_{\Psi_{\tau}^{FL}(x)}, \quad \dot{x}_{\tau}^2 \equiv g_{AB} \dot{x}_{\tau}^A \dot{x}_{\tau}^B \Big|_{\Psi_{\tau}^{St}(x)} = a^2 \left(\dot{r}_{\tau}^2 + \Sigma^2 \dot{\theta}_{\tau}^2 \right) \Big|_{\Psi_{\tau}^{FL}(x)},$$

where the dot means derivative with respect to λ . Regarding the second set of conditions (7), we must first identify the normal planes to the two-surface S_{τ} in each spacetime and then construct the most general linear isometry between them. The vector space $(T_x S_{\tau}^{St})_{\perp}$ is spanned by the two mutually orthogonal one-forms

$$dT \Big|_{\Phi_{\tau}^{St}(x)}, \quad \boldsymbol{\kappa} \Big|_{\Phi_{\tau}^{St}(x)} \equiv \epsilon_{AB} \dot{x}_{\tau}^A dx^B \Big|_{\Phi_{\tau}^{St}(x)},$$

where ϵ_{AB} is the totally antisymmetric symbol with $\epsilon_{12} = 1$. Similarly, the vector space $(T_x S_{\tau}^{FL})_{\perp}$ is spanned by the two mutually orthogonal one-forms

$$dt \Big|_{\Phi_{\tau}^{FL}(x)}, \quad \boldsymbol{\alpha} \Big|_{\Phi_{\tau}^{FL}(x)} \equiv \dot{\theta}_{\tau} dr - \dot{r}_{\tau} d\theta \Big|_{\Phi_{\tau}^{FL}(x)}.$$

Thus, the most general linear and isometric application $\xi_{\tau}^x : (T_x S_{\tau}^{FL})_{\perp} \longrightarrow (T_x S_{\tau}^{St})_{\perp}$ is given by the linear extension of

$$\begin{aligned}\xi_{\tau}^x \left(dt \Big|_{\Psi_{\tau}^{FL}(x)} \right) &= \eta_1 F \cosh \beta dT + \eta_2 \sqrt{\frac{G}{\dot{x}_{\tau}^2}} \sinh \beta \boldsymbol{\kappa} \Big|_{\Psi_{\tau}^{St}(x)}, \\ \xi_{\tau}^x \left(\frac{a^2 \Sigma}{\sqrt{\dot{y}_{\tau}^2}} \boldsymbol{\alpha} \Big|_{\Psi_{\tau}^{FL}(x)} \right) &= \eta_1 F \sinh \beta dT + \eta_2 \sqrt{\frac{G}{\dot{x}_{\tau}^2}} \cosh \beta \boldsymbol{\kappa} \Big|_{\Psi_{\tau}^{St}(x)},\end{aligned}$$

where $\beta(\lambda, \phi)$ is an arbitrary function, η_1 and η_2 are arbitrary signs and we have defined $\dot{y}_{\tau}^2 \equiv a^2 \left(\dot{r}_{\tau}^2 + \Sigma^2 \dot{\theta}_{\tau}^2 \right) \Big|_{\Phi_{\tau}^{FL}(x)}$ and $G \equiv \det(g_{AB})$. A consequence of the staticity of the interior region is that $\mathbf{K}_{S_{\tau}}^{St} \left(dT \Big|_{\Phi_{\tau}^{FL}(x)} \right) = 0$. Consequently, the second set of matching conditions (7) becomes

$$\left. \begin{aligned}\mathbf{K}_{S_{\tau}}^{FL} \left(dt \Big|_{\Psi_{\tau}^{FL}(x)} \right) &= \eta_2 \sqrt{\frac{G}{\dot{x}_{\tau}^2}} \sinh \beta \mathbf{K}_{S_{\tau}}^{St} \left(\boldsymbol{\kappa} \Big|_{\Psi_{\tau}^{St}(x)} \right) \\ \frac{a^2 \Sigma}{\sqrt{\dot{y}_{\tau}^2}} \mathbf{K}_{S_{\tau}}^{FL} \left(\boldsymbol{\alpha} \Big|_{\Psi_{\tau}^{FL}(x)} \right) &= \eta_2 \sqrt{\frac{G}{\dot{x}_{\tau}^2}} \cosh \beta \mathbf{K}_{S_{\tau}}^{St} \left(\boldsymbol{\kappa} \Big|_{\Psi_{\tau}^{St}(x)} \right)\end{aligned} \right\}$$

which implies

$$\mathbf{K}_{S_{\tau}}^{FL} \left(dt \Big|_{\Psi_{\tau}^{FL}(x)} \right) = \frac{a^2 \Sigma}{\sqrt{\dot{y}_{\tau}^2}} \tanh \beta \mathbf{K}_{S_{\tau}}^{FL} \left(\boldsymbol{\alpha} \Big|_{\Psi_{\tau}^{FL}(x)} \right).\tag{16}$$

This equation involves only objects from the FL background metric. A trivial calculation shows that $\mathbf{K}_{S_{\tau}}^{FL} \left(dt \Big|_{\Psi_{\tau}^{FL}(x)} \right)$ and $\mathbf{K}_{S_{\tau}}^{FL} \left(\boldsymbol{\alpha} \Big|_{\Psi_{\tau}^{FL}(x)} \right)$ are both diagonal and therefore (16) implies two conditions. Combining them so that the hyperbolic angle β disappears, we obtain the equation

$$-\dot{\theta}_\tau \ddot{r}_\tau + \dot{r}_\tau \ddot{\theta}_\tau + \frac{\Sigma_{,r}}{\Sigma} \dot{r}_\tau^2 \dot{\theta}_\tau + \left(\frac{\dot{r}_\tau^2}{\Sigma^2} + \dot{\theta}_\tau^2 \right) \frac{\cos \theta}{\sin \theta} \dot{r}_\tau \Big|_{S_\tau^{FL}} = 0, \quad (17)$$

where we have dropped a global factor $a_{,t}$ (comma means, as usual, partial derivative) which is non-zero because the cosmological model is assumed to be expanding (our aim is precisely analyzing the possible effect of the cosmological expansion in the local physics). (17) is an ordinary differential equation which defines the possible shapes of S_τ^{FL} . It can be explicitly solved to give

$$\Sigma(r, \epsilon) \Big|_{r=r_\tau(\lambda)} = \frac{2\sigma_1 \left(\sigma_0 \cos \theta + \sqrt{1 - \sigma_0^2 \sin^2 \theta} \right)}{\left(\sigma_0 \cos \theta + \sqrt{1 - \sigma_0^2 \sin^2 \theta} \right)^2 + \epsilon \sigma_1^2} \Big|_{\theta=\theta_\tau(\lambda)}, \quad (18)$$

where σ_1 and σ_0 are arbitrary integration constants (the particular solution $\theta = \pi/2$ can be obtained from this expression by putting $\sigma_0 = 1$ and $\sigma_1 = 0$ and following an appropriate limiting procedure). In order to identify this surface, let us analyze its intrinsic geometry. Using the defining relation (18) it is not difficult to obtain the first fundamental form in S_τ^{FL} , which reads

$$ds_{S_\tau^{FL}}^2 = \frac{\Sigma^2}{1 - \sigma_0^2 \sin^2 \theta} d\theta^2 + \Sigma^2 \sin^2 \theta d\phi^2 \Big|_{r=r_\tau(\theta)}.$$

Evaluating the curvature scalar $R_{S_\tau^{FL}}$ for this metric we find

$$R_{S_\tau^{FL}} = \frac{[(1 + \sigma_0)^2 + \epsilon \sigma_1^2][(1 - \sigma_0)^2 + \epsilon \sigma_1^2]}{2\sigma_1^2}, \quad (19)$$

which is constant throughout S_τ^{FL} . Thus, this two-surface is maximally symmetric and hence [13] it is either a two-sphere, a two-plane or a hyperbolic plane depending on whether the sign of $R_{S_\tau^{FL}}$ is positive, zero or negative, respectively. Due to (19), it follows that in a closed FL spacetime ($\epsilon = 1$) the matching two-surface S_τ^{FL} must be a two-sphere, in a flat FL cosmology ($\epsilon = 0$) both two-spheres and two-planes are possible and in an open FL cosmology ($\epsilon = -1$) the three possibilities above are, in principle, allowed. The physical problem we are investigating is the existence of axially symmetric regions immersed in a Friedman-Lemaître background. When the surfaces S_τ^{FL} are two-planes or hyperbolic planes, the static regions are spatially unbounded (they extend to the spatial infinity in the FL cosmology) and the only possible construction for having a static region in a FL background would be performing a double matching, so that the static region is “sandwiched” between two FL spacetimes (i.e. the static region would have finite width but would be unbounded in the other two spatial directions). Even in that case, the static region cannot be prevented to disconnect the FL background. Thus, these two cases are not physically interesting for the problem we are considering in this paper. Consequently, we will restrict the analysis to the case in which $\{S_\tau^{FL}\}$ are two-spheres (i.e (19) is always positive).

Let us now locate these two-spheres within the FL spacetime. To that end, we take advantage of the high degree of symmetry in the cosmological background. The uniparametric group of transformations generated by the Killing vector

$$\vec{k} = \cos \theta \partial_r - \frac{\Sigma_{,r}}{\Sigma} \sin \theta \partial_\theta \quad (20)$$

is the most general translational isometry leaving the symmetry axis of ∂_ϕ invariant. Let us use the natural parametrization (which will be denoted by γ) induced by (20) in this group of transformations. It is not difficult to show that the transformation $\gamma = \gamma_0$, where γ_0 is given by

$$\Sigma(\gamma_0, \epsilon) = \frac{2\sigma_1 \sigma_0}{\sqrt{[(1 + \sigma_0)^2 + \epsilon \sigma_1^2][(1 - \sigma_0)^2 + \epsilon \sigma_1^2]}},$$

transforms the surface (18) into the two-sphere $r = r_0$, with r_0 given by

$$\Sigma(r_0, \epsilon) = \frac{2\sigma_1}{\sqrt{\left[(1 + \sigma_0)^2 + \epsilon\sigma_1^2\right]\left[(1 - \sigma_0)^2 + \epsilon\sigma_1^2\right]}}.$$

Thus, the matching hypersurface in the FL cosmology can be, at most, a two-sphere moving arbitrarily along the axis of symmetry and with a radius which can change freely as time evolves. At this stage, the result in [8] stating that no cylindrically symmetric Einstein-Straus models can exist, has been recovered and generalized. In those models the matching hypersurface consists of a collection of cylinders with arbitrarily varying radius. The results above show that any model in which the matching hypersurface at any instant of cosmic time is not a two-sphere, a two-plane or a hyperbolic plane is *impossible*.

In order to complete the matching procedure, it is convenient to introduce new coordinate systems in the FL and static spacetimes so that the form of the matching hypersurfaces becomes simpler. To that end, we choose spherical coordinates on each hypersurface $t = \tau$ in the FL spacetime so that the two-sphere S_τ^{FL} is centred at the origin. Completing these coordinates with the global time t , we obtain a well-defined system of coordinates. In order to obtain the line-element in the new coordinate system, we first change the original coordinates (10) into cylindrical coordinates $\{t, \rho, z, \phi\}$ by the standard transformation

$$\Sigma(\rho, \epsilon) = \Sigma(r, \epsilon) \sin \theta, \quad \frac{\Sigma(z, \epsilon)}{\Sigma_{,z}(z, \epsilon)} = \frac{\Sigma(r, \epsilon)}{\Sigma_{,r}(r, \epsilon)} \cos \theta. \quad (21)$$

(t and ϕ remain unchanged). In cylindrical coordinates, the Killing vector (20) reads simply $\vec{k} = \partial_z$ and the corresponding uniparametric group of transformations takes the trivial form

$$t \rightarrow t, \quad \rho \rightarrow \rho, \quad z \rightarrow z + \gamma_0(t), \quad \phi \rightarrow \phi, \quad (22)$$

where γ_0 is allowed to change in time because the centre of the two-sphere S_τ^{FL} can move along the symmetry axis. Applying (22) and then transforming back into spherical coordinates by using (21), we obtain the line-element

$$ds_{FL}^2 = -dt^2 + a^2(t) \left[\left(dr + f(t) \cos \theta dt \right)^2 + \left(\Sigma(r, \epsilon) d\theta - f(t) \Sigma_{,r}(r, \epsilon) \sin \theta dt \right)^2 + \Sigma^2(r, \epsilon) \sin^2 \theta d\phi^2 \right]. \quad (23)$$

where $f(t) \equiv \gamma_0(t)_{,t}$ denotes the motion of the centre of the two-sphere along the symmetry axis and, for notational simplicity, the same symbols $\{t, r, \theta, \phi\}$ are used to name the new coordinates (which will be used from now on). The embedding defining the matching surface takes now the very simple form

$$\begin{aligned} \Phi^{FL} : \Omega &\longrightarrow V^{FL} \\ \Phi^{FL} : \{\tau, \theta, \varphi\} &\longrightarrow \{t = \tau, r = r(\tau), \theta = \vartheta, \phi = \varphi\}, \end{aligned} \quad (24)$$

and the embedding $\Phi_\tau^{FL} : S_\tau \rightarrow V^{FL}$ is obtained simply by fixing $\tau = \text{const.}$ above.

A convenient coordinate system in the static region can be constructed as follows. Due to the staticity of the spacetime, the integral lines of the static Killing ∂_T define a canonical diffeomorphism between different constant time hypersurfaces [14]. Thus, there exists a natural projection

$$\Pi : V^{St} \longrightarrow \Xi,$$

where Ξ is the quotient space of V^{St} defined by the orbits of the static Killing vector (it is, therefore, diffeomorphic to any of the hypersurfaces $\{T = \text{const.}\}$). Finding a coordinate system in an open subset $\mathcal{U} \subset \Xi$ amounts to finding a coordinate system in the four-dimensional open set $\Pi^{-1}(\mathcal{U})$

because the static time T can be used as the fourth coordinate. Let us now consider the projection along Π of the two-spheres $\{S_\tau^{St}\}$. Since S_τ^{St} is completely contained in the hypersurface $\{T = T(\tau)\}$ it follows that S_τ^{St} is isometric to $\Pi(S_\tau^{St})$ which, consequently, must be a two-sphere in Ξ . Furthermore, Ξ is axially symmetric (because of the axial symmetry of V^{St}) and each $\Pi(S_\tau^{St})$ is centred on the symmetry axis of Ξ (this follows from the fact that $\{S_\tau^{St}\}$ are invariant under the action of the axial symmetry in V^{St}). Defining the subset of Ξ

$$\mathcal{W} = \bigcup_{\tau \in J} \Pi(S_\tau^{St}),$$

where J is the interval of variation of the parameter τ , it is clear that either \mathcal{W} is also a two-sphere or \mathcal{W} has a non-empty interior. The first situation is marginal and it only happens when all the two-spheres $\{S_\tau^{St}\}$ are projected to each other by the integral curves of ∂_T . Consequently, they must all have the same radius, which translates into $\Sigma(r(\tau))a(\tau) = \text{const.}$ in the FL background. This fixes $r(\tau)$ once the scale factor is given. Furthermore, a trivial calculation shows that the matching conditions (1) imply $f \equiv 0$. Thus, the matching hypersurface in the FL spacetime is completely fixed and must be a two-sphere of constant geometrical radius with its centre being at rest with respect to the cosmological flow. Hence, the shape of the static cavity *must* be spherically symmetric in this case.

In the generic situation, the set of matching two-spheres $\{S_\tau^{St}\}$ are not all projected to each other by the integral curves of ∂_T and, hence, \mathcal{W} has a non-empty interior $\overset{\circ}{\mathcal{W}}$. At any point $p \in \overset{\circ}{\mathcal{W}}$ there exists an open neighbourhood $U_p \subset \overset{\circ}{\mathcal{W}}$ and a sufficiently small open subinterval $I_p \subset J$ such that the set of two-spheres $\{\Pi(S_\tau^{St}), \tau \in I_p\}$ foliates U_p . In other words, for any point $q \in U_p$ there exists a single value of $\tau_q \in I_p$ such that q belongs to $\Pi(S_{\tau_q}^{St})$. A coordinate system in U_p can now be constructed as follows. First, choose the natural angular coordinates $\{\theta, \hat{\phi}\}$ on each two-sphere $\Pi(S_\tau^{St})$ such that $\theta = 0$ and $\theta = \pi$ correspond to the axial symmetry axis of Ξ (this choice is possible because each two-sphere S_τ^{St} is centred with respect to the axis of symmetry in V^{St}). The third coordinate necessary to identify any point $q \in U_p \subset \Xi$ can be chosen as the unique value $\tau_q \in I_p$ identifying the two-sphere passing through q . In this way, a coordinate system around any point $p \in \overset{\circ}{\mathcal{W}}$ is constructed. This coordinate system cannot, in general, cover the whole set $\overset{\circ}{\mathcal{W}}$ (because two different two-spheres with arbitrarily close values of τ may intersect). Due to the purely local character of the matching conditions, this is not an obstacle to proceed with the matching. In order to simplify the matching equations and the final form of the interior static metric obtained below, it turns out to be convenient to use the symbol t to denote the coordinate identifying the two-sphere passing through any point $q \in U_p$. This coordinate is spacelike by construction and it has nothing to do with the cosmic time in the FL background so that they should not be confused. The meaning of t should become clear from the context. The reason why the same symbol is used for such different objects is that both take the same value at every point on the matching hypersurface, which allows for an important simplification in the writing of the expressions below.

The line-element for the static spacetime in the coordinate system just constructed takes the form

$$ds_{St}^2 = -F^2(t, \theta) dT^2 + Q^2(t, \theta) dt^2 + \hat{r}^2(t) \left[\left(d\theta + H(t, \theta) dt \right)^2 + \sin^2 \theta d\hat{\phi}^2 \right], \quad (25)$$

which is the most general metric for a static, axially symmetric and orthogonally transitive spacetime such that the two-surfaces $T = \text{const.}$, $t = \text{const.}$ are two-spheres.

The embedding of the matching hypersurface Ω into V^{St} takes now the simple form

$$\begin{aligned} \Phi^{St} : \Omega &\longrightarrow V^{St} \\ \Phi^{St} : \{\tau, \vartheta, \varphi\} &\longrightarrow \{T = T(\tau), t = \tau, \theta = \vartheta, \hat{\phi} = \varphi\} \end{aligned} \quad (26)$$

and the corresponding embeddings $\Phi_\tau^{St} : S_\tau \rightarrow V^{St}$ are obtained by fixing $\tau = \text{const.}$

We are now in a favourable position to complete the matching problem. We already made use of the constraint conditions (5) and (7) in order to restrict the form of Ω . However, not all their information has been extracted yet. The equality of the two first fundamental forms on S_τ gives now the single relation

$$\hat{r}(t) = a(t)\Sigma(r, \epsilon) \Big|_{r=r(t)}. \quad (27)$$

The vector spaces $(T_x S_\tau^{FL})_\perp$ and $(T_x S_\tau^{St})_\perp$ are

$$(T_x S_\tau^{FL})_\perp = \left\langle dt|_{\Psi_\tau^{FL}(x)}, dr|_{\Psi_\tau^{FL}(x)} \right\rangle, \quad (T_x S_\tau^{St})_\perp = \left\langle dT|_{\Psi_\tau^{St}(x)}, dt|_{\Psi_\tau^{St}(x)} \right\rangle$$

and the most general linear isometry ξ_τ^x between them is the linear extension of

$$\begin{aligned} \xi_\tau^x \left(dt|_{\Psi_\tau^{FL}(x)} \right) &= \eta_1 \cosh \beta F dt + \eta_2 \sinh \beta Q dt \Big|_{\Psi_\tau^{St}(x)}, \\ \xi_\tau^x \left(dr|_{\Psi_\tau^{FL}(x)} \right) &= \eta_1 F \left(\frac{\sinh \beta}{a} - f \cos \theta \cosh \beta \right) dt + \\ &\quad + \eta_2 Q \left(\frac{\cosh \beta}{a} - f \cos \theta \sinh \beta \right) dt \Big|_{\Psi_\tau^{St}(x)}. \end{aligned} \quad (28)$$

A straightforward calculation shows that (7) are equivalent to the three equations

$$\begin{aligned} \sin \theta H_{,\theta} - \cos \theta H|_{S_\tau} &= 0, \\ \eta_2 \hat{r} \sinh \beta (H \hat{r} \cos \theta - \hat{r}_{,t} \sin \theta) - Q \Sigma^2 a_{,t} \sin \theta|_{S_\tau} &= 0, \\ \Sigma a_{,t} \cosh \beta + \Sigma_{,r} \sinh \beta|_{S_\tau} &= 0, \end{aligned}$$

which can be trivially solved to give

$$H(t, \theta) = L(t) \sin \theta, \quad Q(t, \theta) = \frac{\hat{r}_{,t} - L \hat{r} \cos \theta}{\eta_2 \sqrt{\Sigma_{,r}^2 - \Sigma^2 a_{,t}^2}} \Big|_{r=r(t)}, \quad (29)$$

$$\sinh \beta = \frac{-\Sigma a_{,t}}{\sqrt{\Sigma_{,r}^2 - \Sigma^2 a_{,t}^2}} \Big|_{r=r(t)}, \quad \cosh \beta = \frac{\Sigma_{,r}}{\sqrt{\Sigma_{,r}^2 - \Sigma^2 a_{,t}^2}} \Big|_{r=r(t)}, \quad (30)$$

where $L(t)$ is an arbitrary function. Thus, the θ dependence of the two metric coefficients Q and H in (25) is completely fixed. These expressions show that a necessary condition for the existence of the matching is

$$\Sigma_{,r}^2 - \Sigma^2 a_{,t}^2 \Big|_{r=r(t)} > 0. \quad (31)$$

This inequality can be geometrically interpreted by noticing that the two-sphere S_t^{FL} is non-trapped (see e.g. [15] for a definition) if and only if (31) is fulfilled. Thus, it follows that a trapped (or marginally trapped) surface cannot be the boundary of an axially symmetric static region, which is an intuitively clear result. Using the expression (43) below for the cosmic energy-density ρ^{FL} , the inequality (31) can be rewritten as

$$\Sigma(r(t), \epsilon) < \frac{1}{a(t)} \sqrt{\frac{3}{\rho^{FL}(t)}}. \quad (32)$$

Therefore, the radius of the static region must be smaller than some maximum radius. In particular, this forbids situations in which the radius of the two-sphere S_τ^{FL} grows unboundedly until reaching

infinity (so that, in fact, it becomes a two-plane). Hence, if the matching hypersurface is a two-sphere at any instant of time, it must remain a two-sphere for ever (of course, this two-sphere can become a point and then disappear).

Having exhausted the constraint matching conditions, we must now consider the more complicated set of conditions (1) and (3). The task is, however, much simpler now because of all the information we already know. The equality of the two first fundamental forms on Ω provides

$$L = - \frac{\Sigma_{,r}}{\Sigma} \Big|_{r=r(t)} f(t), \quad FT_{,t} = \frac{\Sigma_{,r} + aa_{,t}\Sigma(r_{,t} + f \cos \theta)}{\sqrt{\Sigma_{,r}^2 - a_{,t}^2\Sigma^2}} \Big|_{r=r(t)}. \quad (33)$$

which fixes the θ dependence of the remaining metric coefficient F in the static spacetime. Using (27), (29) and (33), the static line-element can be rewritten in the form

$$ds_{St}^2 = - \frac{[\Sigma_{,r} + aa_{,t}\Sigma(r_{,t} + f \cos \theta)]^2}{T_{,t}^2(\Sigma_{,r}^2 - a_{,t}^2\Sigma^2)} dt^2 + \frac{[a_{,t}\Sigma + a\Sigma_{,r}(r_{,t} + f \cos \theta)]^2}{\Sigma_{,r}^2 - a_{,t}^2\Sigma^2} dt^2 + \\ + a^2\Sigma^2 \left(d\theta - \frac{\Sigma_{,r}}{\Sigma} f \sin \theta dt \right)^2 + a^2\Sigma^2 \sin^2 \theta d\phi^2 \Big|_{r=r(t)}, \quad (34)$$

which is completely given in terms of cosmological quantities, except for the function $T(t)$ (which fixes the form of the matching hypersurface as seen from the static region). In order to impose the second set of matching conditions we choose the rigging

$$\vec{l}_{FL} = \frac{1}{a} \partial_r \implies \\ l_{FL} \Big|_{\Omega^{FL}} = a \left(f \cos \theta dt + dr \right) \Big|_{\Omega^{FL}}, \quad l_{St} \Big|_{\Omega^{St}} = \eta_1 F \sinh \beta dt + \eta_2 Q \cosh \beta dt \Big|_{\Omega^{St}},$$

where the one-form l_{St} has been obtained from l_{FL} by applying the linear isometry (28). Evaluating (3) involves now a rather long calculation which was performed with the aid of the computer algebra program REDUCE [16]. Only the (τ, τ) component is not identically satisfied and it provides two conditions. Whenever f is not identically vanishing these two equations can be rewritten as

$$T_{,tt}f = T_{,t} \left[f_{,t} + 2f \frac{a_{,tt}\Sigma_{,r}^2 + a_{,t}\Sigma\Sigma_{,rr,t}(\epsilon + a_{,t}^2)}{a_{,t}(\Sigma_{,r}^2 - a_{,t}^2\Sigma^2)} \Big|_{r=r(t)} \right], \quad (35)$$

$$(aa_{,t}\Sigma r_{,t} + \Sigma_{,r})a_{,t}f_{,t} + aa_{,t}^2\Sigma_{,r}f^3 + \\ + [-aa_{,t}^2(\Sigma r_{,tt} + \Sigma_{,r}r_{,t}^2) + 2(a_{,tt}\Sigma_{,r} + \epsilon a_{,t}\Sigma r_{,t})]f \Big|_{r=r(t)} = 0. \quad (36)$$

The case $f \equiv 0$ can be obtained from these equations by substituting $f_{,t}$ from (36) into (35), then dropping a global factor f and finally putting $f = 0$. Only one equation survives in that particular case.

The expressions (35) and (36) are differential equations for $T(t)$ and $f(t)$. Thus, they determine the form of the matching hypersurface in the static spacetime and the motion of the centre of the two spheres $\{S_t^{FL}\}$ along the axis of symmetry in the FL spacetime. Both equations can be explicitly integrated once the scale factor $a(t)$ and the evolution $r(t)$ of the radius of the two-spheres $\{S_t^{FL}\}$ are given. In order to solve (36) two different cases must be considered. In the particular case

$$\Sigma_{,r} + aa_{,t}\Sigma r_{,t} \Big|_{r=r(t)} \equiv 0, \quad (37)$$

the equation becomes algebraic and its solution is (the other solution $f \equiv 0$ is impossible due to (33))

$$f^2(t) = \frac{a_{,t}^2 + \epsilon - aa_{,tt}}{a^2 a_{,t}^2} = \frac{(\rho + p)^{FL}}{2a_{,t}^2},$$

where the last equality follows from the standard expressions for the energy-density and pressure in the FL spacetime in terms of the scale factor (see (43) below).

In the generic case, (37) does not hold and (36) is a Bernouilli differential equation which can be integrated [17] in terms of quadratures as

$$f(t) = \frac{\alpha e^{\int f_1 dt}}{\sqrt{1 - 2\alpha^2 \int f_2 e^{2\int f_1 dt} dt}}, \quad (38)$$

where α is an arbitrary constant and $f_1(t)$, $f_2(t)$ are defined as

$$f_1(t) \equiv \frac{aa_{,t}^2 (\Sigma r_{,tt} + \Sigma_{,r} r_{,t}^2) - 2 (a_{,tt} \Sigma_{,r} + \epsilon a_{,t} \Sigma r_{,t})}{a_{,t} (aa_{,t} \Sigma r_{,t} + \Sigma_{,r})} \Big|_{r=r(t)}, \quad f_2(t) \equiv \frac{-aa_{,t} \Sigma_{,r}}{aa_{,t} \Sigma r_{,t} + \Sigma_{,r}} \Big|_{r=r(t)}.$$

Regarding (35), this is a linear differential equation for $T_{,t}$ which is therefore trivial to integrate. The structure of its general solution is $T(t) = \alpha_1 T_0(t) + \alpha_0$, where $T_0(t)$ is any particular solution and α_0 , α_1 are integration constants. These constants are irrelevant as the form of the matching hypersurface remains unaffected if they change. They simply correspond to the freedom (13) in defining the global static time in any static spacetime. Thus, α_0 and α_1 can be fixed arbitrarily without loss of generality.

Thus, we conclude that the matching of a Friedman-Lemaître spacetime with an axially symmetric static region allows for an arbitrary scale factor in the FL spacetime and for an arbitrary function $r(t)$ describing the evolution of the radius of the matching two-spheres (as long as the inequality (32) is fulfilled). Then, the matching conditions fix the motion of the centre of the two-spheres along the axis of symmetry through (38). They also fix uniquely the form of the matching hypersurface in the interior static region *and* the full geometry of the static spacetime (in an open region around the matching hypersurface), which is given by the line-element (34). In the next section, the geometry of the interior static geometry will be analyzed.

IV. GEOMETRY OF THE STATIC SPACETIME.

Let us first of all discuss the domain of applicability of the coordinate system $\{T, t, \theta, \hat{\phi}\}$ in the static spacetime. As remarked in the previous section, these coordinates are well-defined at least in a sufficiently small open neighbourhood around any point in the interior of $\Pi^{-1}(\Pi(\Omega^{St}))$. A priori, there is no reason to expect this coordinate system to cover the whole open set $\Pi^{-1}(\overset{\circ}{\mathcal{W}})$. However, this turns out to be the case and a single coordinate patch can be used to describe all the relevant interior region. Before showing this explicitly, it is worth pointing out that the form of the interior static metric at points *not* belonging to $\Pi^{-1}(\overset{\circ}{\mathcal{W}})$ can not be determined at all by the matching conditions. This is because the matching procedure is purely local and there is no way of relating points outside this open set with the matching hypersurface. Determining the interior metric at those points requires further information like analyticity conditions or the Einstein field equations for some type of energy-momentum contents. Thus, all the discussion below concerning the geometry of the static region is restricted to $\Pi^{-1}(\overset{\circ}{\mathcal{W}})$.

In order to see explicitly that $\{T, t, \theta, \hat{\phi}\}$ is a valid coordinate system everywhere, we must show that no coordinate singularities are present in the line-element (34) or, equivalently, we must proof that any possible coordinate singularity is, in fact, an essential singularity. The metric (34) becomes singular at any point satisfying

$$\Sigma_{,r} + aa_{,t} \Sigma (r_{,t} + f \cos \theta) = 0 \quad \text{or} \quad a_{,t} \Sigma + a \Sigma_{,r} (r_{,t} + f \cos \theta) = 0. \quad (39)$$

The remaining possibilities, like $T_{,t} = 0$ or $f \rightarrow +\infty$, have clear implications for the matching hypersurface as seen from the FL spacetime (where the coordinates have a transparent meaning) and can be discarded as unphysical for the problem we are considering. In order to show that points

satisfying (39) are essential singularities, let us write down the energy-momentum tensor of the static spacetime. Using the orthonormal tetrad

$$\begin{aligned}\theta^0 &= \frac{\Sigma_{,r} + aa_{,t}\Sigma(r_{,t} + f \cos \theta)}{T_{,t}\sqrt{\Sigma_{,r}^2 - a_{,t}^2\Sigma^2}}dT \Big|_{r=r(t)}, \quad \theta^1 = \frac{a_{,t}\Sigma + a\Sigma_{,r}(r_{,t} + f \cos \theta)}{\sqrt{\Sigma_{,r}^2 - a_{,t}^2\Sigma^2}}dt \Big|_{r=r(t)}, \\ \theta^2 &= a\Sigma \left(d\theta - \frac{\Sigma_{,r}}{\Sigma} f \sin \theta dt \right) \Big|_{r=r(t)}, \quad \theta^3 = a\Sigma \sin \theta d\phi \Big|_{r=r(t)}\end{aligned}\quad (40)$$

to evaluate all tensor components, it turns out that the energy-momentum tensor is diagonal and with the structure $(T_{\alpha\beta}) = \text{diag}(\rho, p_r, p_T, p_T)$, where ρ stands for the energy-density, p_r is the radial pressure and p_T the tangential pressure. We borrow the terms radial and tangential pressure from the spherically symmetric case even though the static metric does not possess this exact symmetry in general. However, the static metric is nearly spherically symmetric (its departure being governed by the single function f) since at every point we have a two-sphere passing through it. The structure of the energy-momentum tensor is exactly the same as in the exact spherically symmetric case. This resemblance goes even further because the evaluation of the Weyl spinor in the null tetrad associated canonically with (40) shows that Ψ_2 is the only non-vanishing component, exactly as in the spherically symmetric case. Thus, the static spacetime is algebraically special of Petrov type D (or conformally flat when $\Psi_2=0$).

The explicit expressions for ρ and p_r can be read from the expressions

$$\rho = \frac{3a\Sigma_{,r}(a_{,t}^2 + \epsilon)(f \cos \theta + r_{,t}) + a_{,t}\Sigma(2aa_{,tt} + a_{,t}^2 + \epsilon)}{a^2[a_{,t}\Sigma + a\Sigma_{,r}(r_{,t} + f \cos \theta)]} \Big|_{r=r(t)}, \quad (41)$$

$$\rho + p_r = \frac{a(\rho + p)^{FL}(\Sigma_{,r}^2 - a_{,t}^2\Sigma^2)(f \cos \theta + r_{,t})}{[a_{,t}\Sigma + a\Sigma_{,r}(r_{,t} + f \cos \theta)][\Sigma_{,r} + aa_{,t}\Sigma(r_{,t} + f \cos \theta)]} \Big|_{r=r(t)}, \quad (42)$$

where ρ^{FL} and p^{FL} stand for the cosmic energy-density and pressure in terms of the scale factor

$$\rho^{FL} = \frac{3(a_{,t}^2 + \epsilon)}{a^2}, \quad p^{FL} = \frac{-2aa_{,tt} - a_{,t}^2 - \epsilon}{a^2}. \quad (43)$$

Regarding p_T , its expression takes different forms depending on whether $aa_{,t}\Sigma r_{,t} + \Sigma_{,r} \neq 0$ or not. In the first case we obtain

$$\begin{aligned}\rho + p_T &= \frac{(\rho + p)^{FL}a\Sigma\Sigma_{,r} + 2(\rho + p)^{FL}a\Sigma_{,r}^2(r_{,t} + f \cos \theta)}{2[a_{,t}\Sigma + a\Sigma_{,r}(r_{,t} + f \cos \theta)][\Sigma_{,r} + aa_{,t}\Sigma(r_{,t} + f \cos \theta)]} + (\rho + p)^{FL}\Sigma \times \\ &\times \frac{a\Sigma\Sigma_{,r}r_{,t}(3a_{,t}^2 - 2aa_{,tt}) + a_{,t}[\Sigma_{,r}^2(3 + a^2f^2) - a^2(1 + \epsilon\Sigma^2)r_{,t}^2 - a^2\Sigma\Sigma_{,r}r_{,tt}]}{2[a_{,t}\Sigma + a\Sigma_{,r}(r_{,t} + f \cos \theta)][\Sigma_{,r} + aa_{,t}\Sigma(r_{,t} + f \cos \theta)][\Sigma_{,r} + aa_{,t}\Sigma r_{,t}]} \Big|_{r=r(t)}\end{aligned}\quad (44)$$

and in the second case ($aa_{,t}\Sigma r_{,t} + \Sigma_{,r} \equiv 0$), the corresponding expression reads

$$\begin{aligned}\rho + p_T &= \frac{1}{4aa_{,t}\Sigma f \cos \theta (a_{,t}^2\Sigma^2 - \Sigma_{,r}^2 + aa_{,t}\Sigma\Sigma_{,r}f \cos \theta)} \left(aa_{,t}\Sigma^2\Sigma_{,r}(\rho + p)^{FL}_{,t} + \right. \\ &\left. + 2(\rho + p)^{FL}\Sigma_{,r}[2aa_{,t}\Sigma\Sigma_{,r}f \cos \theta + \Sigma^2(3a_{,t}^2 + 4\epsilon - aa_{,tt}) - 2] \right) \Big|_{r=r(t)}.\end{aligned}$$

Since the scalar quantities ρ , p_r and p_T have a clear physical meaning, any divergence in them corresponds to a true singularity in the static metric. It is now clear that at any point satisfying (39) we have divergences in ρ and/or p_r and, therefore, we have a matter singularity rather than just a bad coordinate description. This type of singularities leads to physically unrealistic models and should,

therefore, be discarded (notice that these singularities are “touched” by the matching hypersurface and cannot be interpreted as singular “sources” for the static metric, like in the Schwarzschild spacetime)

The expressions for ρ , p_r and p_T above allow also for obtaining a number of interesting conclusions. When the FL spacetime is de Sitter (or anti-de Sitter) then $(\rho + p)^{FL} = 0$ and the energy-momentum tensor in the static spacetime becomes $T_{\alpha\beta} = \Lambda g_{\alpha\beta}^{St}$ (with the same value for Λ as the cosmic background). Furthermore, the static spacetime can be easily seen to be conformally flat and therefore it must necessarily be de Sitter (anti-de Sitter) [18]. Thus, the following result holds

Proposition 1 *Any static and axially symmetric spatially compact region in a de Sitter background must have the same geometry as the exterior manifold.*

In other words, de Sitter (anti-de Sitter) spacetimes do not admit non-trivial static and axially symmetric spatially compact regions. Thus, we can assume from now on that that the FL spacetime is not de Sitter or anti-de Sitter, i.e. $(\rho + p)^{FL} \neq 0$.

If the static region is imposed to have an energy-momentum tensor with $\rho + p_r = 0$ (this includes the important subcases of vacuum, Λ -term and non-null Einstein-Maxwell solutions, see below), then (42) immediately implies $f = 0$ and $r_{,t} = 0$ ($\iff r(t) = r_0$). Thus, the matching hypersurface must be a two-sphere comoving with the cosmological flow. Conversely, if the matching two-sphere is comoving with the cosmological flow ($f = 0$, $r(t) = r_0$), then (42) implies $\rho + p_r = 0$, which shows the equivalence of the two conditions. In this situation, the differential equation for $T_{,tt}$ (35) can be integrated to give

$$T_{,t} = \left. \frac{\Sigma_{,r}}{\Sigma_{,r}^2 - a_{,t}^2 \Sigma^2} \right|_{r=r_0}$$

(the integration constants are irrelevant, as discussed above) so that the line-element in the static region becomes

$$ds^2 = -(\Sigma_{,r}^2 - a_{,t}^2 \Sigma^2) dT^2 + \left. \frac{a_{,t}^2 \Sigma^2}{\Sigma_{,r}^2 - a_{,t}^2 \Sigma^2} dt^2 + a^2 \Sigma^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right|_{r=r_0}, \quad (45)$$

which is obviously spherically symmetric. Performing the coordinate change

$$a(t)\Sigma(r_0) = \check{r}$$

and defining the standard mass function $m(\check{r})$ [19] as

$$a_{,t}^2 = -\epsilon + \frac{2m(a\Sigma)}{a\Sigma^3},$$

the line element (45) transforms into

$$ds^2 = -\left(1 - \frac{2m(\check{r})}{\check{r}}\right) dT^2 + \frac{d\check{r}^2}{1 - \frac{2m(\check{r})}{\check{r}}} + \check{r}^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (46)$$

It is interesting to notice that this line-element is the most general static and spherically symmetric metric with $\rho + p_r = 0$. Thus, *all* these metrics can be matched with an appropriate exterior FL spacetime through a spherically symmetric matching hypersurface. The energy-momentum contents and the only non-zero Weyl spinor coefficient, Ψ_2 , of this spacetime read

$$\rho = -p_r = \frac{2m_{,\check{r}}}{\check{r}^2}, \quad p_T = -\frac{m_{,\check{r}}\check{r}}{\check{r}}, \quad \Psi_2 = -\frac{m_{,\check{r}}\check{r}}{6\check{r}} + \frac{2m_{,\check{r}}}{3\check{r}^2} - \frac{m}{\check{r}^3}, \quad (47)$$

while the energy-density and pressure in the FL cosmology are

$$\rho^{FL} = \frac{6m}{\check{r}^3}, \quad p^{FL} = -\frac{2m_{,\check{r}}}{\check{r}^2}.$$

Thus, we can conclude

Proposition 2 *The matching between a static and axially symmetric spacetime and a Friedman-Lemaître cosmology across a hypersurface described by a two-sphere comoving with the cosmological flow must have a spherically symmetric interior with metric given by (46) where $m(\tilde{r})$ is an arbitrary function describing the total mass contained in the static region.*

Since vacuum and Λ -term spacetimes are particular cases of $\rho + p_r = 0$, the two following results hold

Corollary 1 *The only Λ -term static and axially symmetric spatially compact region within a FL spacetime must be a two-sphere comoving with the cosmological flow and interior geometry given by (46) with $m = m_0 + \frac{\rho_0 \tilde{r}^3}{6}$ (m_0 and ρ_0 constants).*

Corollary 2 *The only static and axially symmetric, spatially compact, vacuum region in a Friedman-Lemaître spacetime must be a two-sphere comoving with the cosmological flow and with Schwarzschild interior geometry (the Einstein-Straus model)*

Let us next discuss Einstein-Maxwell spacetimes, i.e. spacetimes with an energy-momentum tensor corresponding to an electromagnetic field outside any mass and charge sources. Static spacetimes with a pure null electromagnetic source are impossible [20], so we are left with the non-null case. In this case (and due to the diagonal form of the energy-momentum tensor) the so-called Rainich conditions (which are necessary and sufficient conditions for non-null electrovacuum spacetimes) [18], imply that $(T_{\alpha\beta})$ must take the form

$$T_{\alpha\beta}^{EM} = \text{diag}(\eta, -\eta, \eta, \eta).$$

where η is some positive quantity. Hence, $\rho + p_r = 0$ follows and the metric must take the spherically symmetric form (46). The general Einstein-Maxwell solution for such a metric is the well-known Reissner-Nordström solution, with $m(\tilde{r}) = M - \frac{e^2}{2\tilde{r}}$ (where M is the mass and e is electric charge of the point source). The scale factor and the energy contents in the FL spacetime are then given by

$$a_{,t}^2 = -\epsilon + \frac{2M}{a\Sigma^3} - \frac{e^2}{a^2\Sigma^4} \Big|_{r=r_0}, \quad \rho^{FL} = \frac{6M}{a^3\Sigma^3} - \frac{3e^2}{a^4\Sigma^4} \Big|_{r=r_0}, \quad p^{FL} = -\frac{e^2}{a^4\Sigma^4} \Big|_{r=r_0}.$$

Summarising, we have

Proposition 3 *The only static and axially symmetric electrovacuum region in a Friedman-Lemaître spacetime must be a two-sphere comoving with the cosmological flow and with Reissner-Nordström interior geometry.*

Having discussed vacuum, Λ -term and electromagnetic fields, let us next analyse the perfect-fluid case. Imposing $p_r - p_T = 0$ and using the expressions above, we obtain a linear relation in $\cos\theta$. The vanishing of the coefficient in $\cos\theta$ implies necessarily $f \equiv 0$ while the other coefficient provides a rather long equation relating $a(t)$ and $r(t)$. Thus the following result also holds

Proposition 4 *Any static and axially symmetric perfect-fluid region immersed in a Friedman-Lemaître cosmology must be spherically symmetric and its boundary must be an expanding (or contracting) two-sphere with its centre being at rest with respect to the cosmological flow.*

Here the terms *expanding* and *contracting* are defined with respect to the cosmological fluid flow.

All these results above show that the most commonly used energy-momentum tensors imply necessarily that the matching hypersurface must be a two-sphere with its center being comoving with the cosmological fluid (and hence the matching hypersurface is spherically symmetric). Furthermore, the interior geometry must also be exactly spherically symmetric in the relevant open set $\Pi^{-1}(\overset{\circ}{\mathcal{W}})$ (this set may be empty in some marginal cases - see the previous section- but even in that particular situation the conclusions on the form of the matching hypersurface hold). Consequently, spherical

symmetry is indeed an essential ingredient for the existence of reasonable static cavities in a FL expanding background. This implies that the Einstein-Straus model represents a very isolated situation, which poses serious doubts on the suitability of this model for providing a definitive answer to the problem of the influence of the cosmological expansion on the local physics. Even though this effect is probably very small at solar system scales (there is no experimental evidence for such an effect), this is not so clear at larger scales (ranging from galaxies to clusters of galaxies or even larger). In any case, more adequate models describing the problem should be analyzed before reaching definitive conclusions on this subject.

Before finishing this paper, a final comment may be adequate. Solving any matching problem has always a dual interpretation. Due to the symmetry $(+)\leftrightarrow(-)$ of the matching conditions, solving a matching between two spacetimes produces always as a side effect the matching of the two complementary regions discarded in the original matching (see [21] for a clear discussion of this fact). In our case, the complementary situation corresponds to an interior spatially homogeneous and isotropic perfect fluid (with FL metric) and an static, axially symmetric exterior. Thus, irrespectively of any conditions on the static exterior (for instance, no conditions on asymptotic flatness or similar), it turns out that the shape of the interior fluid must be a two-sphere at each instant of time. Furthermore, for the most relevant exterior metrics (in particular vacuum and electrovacuum), the shape of the autogravitating fluid must be exactly spherically symmetric and the exterior geometry must be also exactly spherically symmetric. This side result is interesting in its own and can be phrased as follows.

A Friedman-Lemaître spacetime acting as a source for static and axially symmetric spacetimes with physically reasonable energy-momentum tensors (when the two canonical times match properly) requires that both the boundary of the fluid and the exterior geometry are exactly spherically symmetric, and hence the global spacetime must also be spherically symmetric.

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